Hull-White Convertible Bond Valuation

A convertible bond pays the holder periodic coupon payments from the issuer, but can also convert it into the issuer's stock. The model uses a two-factor trinomial tree for pricing the convertible bond, where the two factors are

- the short term interest rate and
- the issuer's stock price.

Here we assume that the short term interest rate process follows a stochastic differential equation (SDE) of the Hull-White form, and that the stock's price follows geometric Brownian motion with drift. For tractability, however, the stock's price SDE is modified so that its drift does not depend on the stochastic short-term interest rate.

Based on the Hull-White single-factor tree building approach, respective trinomial trees are constructed for the short-term interest rate and stock's price processes. Using the Hull-White two-factor tree building procedure, a combined tree is constructed by matching the mean, variance and correlation corresponding to each combined tree node. The convertible bond price is given from the combined tree by backward induction.

Here the issue time refers to the coupon payment immediately prior to, or including, the valuation time; otherwise it corresponds to the bond's issuance. Since the valuation time is taken to be zero, the issue time must be less than or equal to zero.

We assume that the short-term interest rate process satisfies a SDE of Hull-White form,

$$dr = (\theta - ar)dt + \sigma dW,$$

where

- θ is a parameter chosen to match the initial term structure of zero coupon bond prices,
- a is a mean reversion parameter,
- σ is the short rate volatility and W is standard Brownian motion.

Furthermore we assume that the stock's price process follows geometric Brownian motion with drift, of the form

$$dS = S(r dt + \sigma dW),$$

Where σ is the stock's price volatility and W is standard Brownian motion.

Here the Brownian motions respectively driving the short-term interest rate and stock's price processes have constant instantaneous correlation coefficient. A drawback, here, is that the SDE above for the stock's price admits only a *zero* dividend yield.

Solving the SDE above for the stock's price yields

$$S = S_0 e^{\int\left(r - \frac{\sigma^2}{2}\right)ds + \sigma W}.$$

Observe that the stock's price depends on the random short-term interest rate. For tractability, then, the stock's price is approximated by

$$\tilde{S} = S_0 E\left(e^{\int\left(r - \frac{\sigma_s^2}{2}\right)ds}\right)e^{\sigma W}.$$

We build respective trinomial trees for the approximate stock's price and short-term interest rate processes above. This construction is based on Hull and White's single factor tree-building technique, but is generalized to accommodate non-uniformly spaced partitions of the interval to bond maturity. The separate trinomial trees are then combined into a joint tree using Hull and White's two-factor tree building procedure.

Here the set of nodes at a particular time slice on the combined tree is given by the cross product of the nodes on each respective tree at this time slice. From each node on the combined tree emanate nine children; the branching probabilities are given by the product of the corresponding probabilities on the respective trees, but shifted by a certain amount so to match the correlation between the random variables associated with the combined tree node.

The convertible bond's price is given by traversing the combined tree using backward induction. At a combined tree node, let

- $Coup$ denote the coupon payment,
- H denote the bond holding value,
- O denote the bond price,
- X denote the stock conversion ratio,
- C denote call strike,
- B denote the put strike, and
- A denote the accrued interest.

If the holding value is greater than the call strike, the bond issuer is assumed to call the bond. The bond holder can then put the bond or convert the bond into stock. Specifically the convertible's price equals

$$O = Coup + \max\left(B + A, SX, \min\left(H, C + A\right)\right).$$

Here the holding value at maturity is given by maximum of the bond principal and the stock price times the conversion ratio, that is,

$$\max(SX, P)$$

where P is the bond principal. The holding value for any other time is the present value of the probability weighted bond prices at each of the node's children. The accrued interest is given by a straight-line interpolation of the coupon amount at the next coupon payment date and zero at the previous coupon payment time.

You can find other convertible bond valuation approach at

https://finpricing.com/lib/EqConvertible.html