Arrear Quanto CMS Valuation

An arrear quanto constant-maturity-swap (CMS) is a swap that pays coupons in a different currency from the notional and in arrears. The underlying swap rate is computed from a forward starting CMS.

Assumes that, under the coupon payment currency (SEK) risk-neutral probability measure, the forward swap rate process corresponding to each swap rate fixing follows Geometric Brownian motion with drift. Each forward swap rate process is then convexity adjusted, and is furthermore expressed under the notional currency (FRF) risk neutral-probability measure by means of a quanto adjustment.

Let the observation times $t_0, \ldots, t_N$, correspond to consecutive quarterly resets, where t_0 corresponds to the start date, and t_N corresponds to the maturity date. Then the seller pays

$$3.95\% \times \Delta t_i$$

at time t_i, for $i = 1, \ldots, N$, where $\Delta t_i = t_i - t_{i-1}$.

At time t_i, where $i = 1, \ldots, N$, we consider a three year CMS, which begins at time t_i and has three payment times, $t_i + j$, where $j = 1, \ldots, 3$. Here the floating side pays the SEKSTIBOR rate,

$$L_{t_i + j - 1}$$

at time $t_i + j$, where $j = 1, \ldots, 3$. The fixed side pays a constant amount at time $t_i + j$, where $j = 1, \ldots, 3$. The swap rate at time t_i, which we denote by s_{t_i}, is the constant fixed payment amount that gives the CMS zero value at time t_i; that is,

$$s_{t_i} = \frac{1 - P_{SEK}(t_i, t_i + 3)}{\sum_{j=1}^{3} P_{SEK}(t_i, t_i + j)}$$

where $P_{SEK}(t, T)$ denotes the price at time t of a Swedish zero coupon bond maturing at time T.

From the above, the holder receives

$$0.99 \times \Delta t_i s_{t_i}$$

at time t_i, for $i = 1, \ldots, N$.

The value of our swap at time zero then equals

$$\text{Notional} \times \left[0.99 \sum_{i=1}^{N} \Delta t_i E\left(s_{t_i}\right) P_{FRF}(0, t_i) - 3.95\% \sum_{i=1}^{N} \Delta t_i P_{FRF}(0, t_i) \right]$$

where

- $P_{FRF}(0, t)$ is the price at time zero of a FRF zero coupon bond maturing at time t,
- E denotes the FRF risk-neutral probability measure, and
- the swap notional is denominated in FRF.

Here we have assumed that the FRF short-term interest rate is deterministic.

We note that the common currency unit in Europe is now taken to be the EURO. Furthermore, the exchange rate from the EURO to an associated currency (e.g., FRF) is fixed, so there is no foreign exchange risk. Therefore, FP London uses a common curve, EURIBOR, for discounting; that is, $P_{FRF}(0,t)$ is replaced by the equivalent discount factor

$$P_{EUR}(0,t),$$

which is the price at time zero of EURO denominated zero coupon bond with maturity of t.

Let y_t^i, for $i = 1,\ldots,N$, denote the forward swap rate at time t for a forward starting SEK CMS, which begins at time t_i and has payments at times $t_i + j$ where $j = 1,\ldots,3$. FP assumes that, under the SEK risk-neutral probability measure, the process $\{y_t^i \mid t \in [0,t_i]\}$ satisfies a stochastic differential equation (SDE) of the form

$$dy_t^i = \sigma_i y_t^i dB_t$$

where

- $\{B_t \mid t \geq 0\}$ is standard Brownian motion, and
- σ_i is the volatility.

Recall that the swap pricing formula,

$$\text{Notional} \times \left[0.99 \sum_{i=1}^{N} \Delta t_i E\left(s_{t_i}\right) P_{FRF}(0,t_i) - 3.95\% \sum_{i=1}^{N} \Delta t_i P_{FRF}(0,t_i) \right],$$

requires the expected swap rate,

$$E\left(s_{t_i}\right), \qquad \text{(A.1)}$$

for $i = 1,\ldots,N$. Since $y_{t_i}^i = s_{t_i}$, FP's approach towards computing (A.1) is to convexity adjust the initial forward swap rate, y_0^i.

To this end let

$$bond(Y;c) = \sum_{i=1}^{3} \frac{c}{(1+Y)^i} + \frac{1}{(1+Y)^3}$$

be the price of a bond, with three year maturity, where

- c is an annually paid coupon value, and
- Y is an annualized yield-to-maturity.

FP's convexity adjusted rate is then given by

$$\hat{y}_0^i = y_0^i - \frac{1}{2}\left(y_0^i\right)^2\left(e^{\sigma_i^2 t} - 1\right) \times \frac{\dfrac{\partial^2\, bond\left(y_0^i; y_0^i\right)}{\partial Y^2}}{\dfrac{\partial\, bond\left(y_0^i; y_0^i\right)}{\partial Y}},$$

The yield to maturity of a bond is the internal rate of return on a bond held until maturity. In other words, it is the discount rate that will provide the investor with a present value V equal to the price of the bond. The yield to maturity does not account for the actual term structure of interest rates: https://finpricing.com/lib/FiBond.html

We wish to express the process $\left\{y_t^i \;\middle|\; t \in [0, t_i]\right\}$ under the FRF risk-neutral probability measure. Let r_t^{SEK} denote the SEK short-term interest rate. Assume that, under the SEK risk-neutral probability measure, the process $\left\{r_t^{SEK} \;\middle|\; t \geq 0\right\}$ satisfies a SDE of the form

$$dr_t^{SEK} = a\left(r_t^{SEK}, t\right)dt + b(t)dB_t$$

where $a(r,t)$ and $b(t)$ are deterministic, sufficiently regular functions.

Let X_t denote the exchange rate from one SEK monetary unit to FRF. Furthermore assume that, under the FRF risk-neutral probability measure, the process $\{X_t \mid t \geq 0\}$ satisfies a SDE of the form

$$dX_t = X_t\left(\left[r_t^{FRF} - r_t^{SEK}\right]dt + \sigma_X dW_t^X\right)$$

where

- r_t^{FRF} is the FRF short-term interest rate, which we assume to be deterministic,
- σ_X is the volatility, and
- $\{W_t^X \mid t \geq 0\}$ is standard Brownian motion.

Then under the FRF risk-neutral probability measure, the process $\{r_t^{SEK} \mid t \geq 0\}$ satisfies the SDE

$$dr_t^{SEK} = \left[a\left(r_t^{SEK}, t\right) - \rho\sigma_X b(t)\right]dt + b(t)dW_t$$

where

- $\{W_t \mid t \geq 0\}$ is standard Brownian motion, and
- ρ is the constant instantaneous correlation coefficient between $\{W_t^X \mid t \geq 0\}$ and $\{W_t \mid t \geq 0\}$.

Observe that, under the SEK risk-neutral probability measure, the forward swap rate process, $\{y_t^i \mid t \in [0, t_i]\}$, is driven by the same Brownian motion, $\{B_t \mid t \geq 0\}$, as the short-term interest rate process, $\{r_t^{SEK} \mid t \geq 0\}$. Then, under the FRF risk-neutral probability measure, the process $\{y_t^i \mid t \in [0, t_i]\}$ satisfies the SDE

$$dy_t^i = y_t^i\left(-\rho\sigma_X\sigma_i dt + \sigma_i dW_t\right).$$